WRITING OF MATHEMATICS, BMATH FIRST YEAR, FINAL EXAMINATION

Please attempt all questions. Total Marks - 100.

Question 1 (28 marks)

(1) Define the notions of a well order and a strict partial order on a set. (3+3 marks)
(2) State the Axiom of Choice, Well Ordering Theorem, Hausdorff Maximality Principle and Zorn's Lemma. (3+3+3+3 marks)
(3) Prove that the Zorn's lemma and Hausdorff Maximality Principle are equivalent. (5+5 marks)

Question 2 (18 marks)

(1) What is meant by saying that two ordered sets are of the "same order type"? (2 marks)
(2) Give examples of two countably infinite well ordered sets $(A, <_A)$ and $(B, <_B)$ which are not of the same order type, and also neither is of the same order type as $(\mathbb{Z}_+, <)$ (the positive integers with the usual "less than" order). Justify your answer. (8 marks)
(3) Let A and B be two well ordered sets. Prove that the dictionary order on $A \times B$ is a well order. (8 marks)

Question 3 (34 marks)

(1) Construct an explicit bijection between $\mathbb{Z}_+ \times \mathbb{Z}_+$ and $\mathbb{Z}_+$. Verify that your map is a bijection. (8 marks)
(2) Prove that a countable union of countable sets is countable. Show that a countable product of countable sets is not necessarily countable. (8+8 marks)
(3) Is the set of all finite subsets of a countable set countable or uncountable? Justify your answer. (10 marks)

Question 4 (20 marks)

(1) What is meant by saying two sets have the same cardinality? (2 marks)
(2) Prove that a set and it's power set do not have the same cardinality. (8 marks)
(3) Let X be the two element set $\{0, 1\}$, and let X^ω denote the product of countably many copies of X. Prove that the set of all countable subsets of X^ω has the same cardinality as the set X^ω. (10 marks)